Q_1: Evaluate each of the following integrals:

1- $\int_0^\infty \sqrt{y}\ e^{-y^3}\ dy$

Sol: let $y^3 = x \ \rightarrow\ x^{\frac{1}{3}} \rightarrow dy = \frac{1}{3} x^{\frac{-2}{3}} dx$ & $as\ y\ \rightarrow 0\ then\ x\ \rightarrow 0\ , as\ y \rightarrow \infty\ then\ x\ \rightarrow \infty$

$\rightarrow$

$\int_0^\infty \sqrt{y}\ e^{-y^2}\ dy = \int_0^\infty \sqrt{x^{\frac{1}{6}}}\ e^{-x} \frac{1}{3} x^{\frac{-2}{3}} dx$

$= \frac{1}{3} \int_0^\infty x^{\frac{-1}{2}}\ e^{-x} dx$

$= \frac{1}{3} \Gamma\left(\frac{1}{2}\right)$

$= \frac{\sqrt{\pi}}{3}$

2- Find $\int_0^1 \frac{dx}{\sqrt{-\ln(x)}}$

Sol:

Let $y = -\ln(x) \rightarrow e^{-y} = x$ and $dx = -e^{-y}$

As $x \rightarrow 0$ then $y \rightarrow\ \infty$ and as $x \rightarrow 1$then $y \rightarrow\ 0$

$\rightarrow \int_0^1 \frac{dx}{\sqrt{-\ln(x)}} = \int_0^\infty \frac{e^{-y}}{\sqrt{y}}\ dy$

$= \int_0^\infty y^{\frac{-1}{2}} e^{-y}\ dy$ Compare with $\int_0^\infty x^{\frac{1}{2}-1} e^{-x} dx$

$= \Gamma\left(\frac{1}{2}\right)$

$=\sqrt{\pi}$

Q_2: Prove that $\beta(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$

Sol:

$$\Gamma(u) = \int_0^\infty t^{u-1} e^{-t} dx$$

Let $t = x^2 \to dt = 2xdx$

$\to \Gamma(u) = \int_0^\infty x^{2u} x^{-2}(2x)e^{-t}dx$

$= 2\int_0^\infty x^{2u-1}e^{-x^2}dx$

And in the same way

$$\Gamma(v) = 2\int_0^\infty y^{2v-1}e^{-y^2}dy$$

Transforming to polar coordinates [$x = \rho cos\emptyset$, y = $\rho sin\emptyset$]

$$\Gamma(u)\Gamma(v) = 4\int_0^{\frac{\pi}{2}}\int_0^\infty \rho^{2(u+v)-1}e^{-\rho^2}cos^{2u-1}(\emptyset)sin^{2v-1}(\emptyset)d\rho d\emptyset$$

$= 4[\int_0^\infty \rho^{2(u+v)-1}e^{-\rho^2}d\rho]$ [$\int_0^{\frac{\pi}{2}} cos^{2u-1}(\emptyset)sin^{2v-1}(\emptyset)d\emptyset$]

$= 2\,\Gamma(u+v)\beta(u,v)$

$$\therefore\ \beta(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$$

Q_3: A/ Find the result of $J_{\frac{5}{2}}(x)$

Sol:

Since $J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x}J_v(x)$ then $J_{v+1}(x) = \frac{2v}{x}J_v(x) - J_{v-1}(x)$

$$\to J_{\frac{5}{2}}(x) = \frac{2\frac{3}{2}}{x}J_{\frac{3}{2}}(x) - J_{\frac{1}{2}}(x)$$

$$J_{\frac{3}{2}}(x) = \frac{2\frac{1}{2}}{x}J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x)$$

$$\to J_{\frac{5}{2}}(x) = \frac{3}{x}\left[\frac{1}{x}J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x)\right] - J_{\frac{1}{2}}(x)$$

$$= \frac{3}{x^2}J_{\frac{1}{2}}(x) - \frac{3}{x}J_{-\frac{1}{2}}(x) - J_{\frac{1}{2}}(x)$$

$$= \left(\frac{3}{x^2} - 1\right)\sqrt{\frac{2}{\pi x}}\sin(x) - \frac{3}{x}\sqrt{\frac{2}{\pi x}}\cos(x)$$

Q_3: B/ Evaluate $\int x\, J_2(\sqrt{x})dx$

Let $y = \sqrt{x} \rightarrow x = y^2$ &$dx = 2ydy \rightarrow$

$\int x\, J_2(\sqrt{x})dx = \int y^2 * 2y\, J_2(y)dy$

$= 2\int y^3\, J_2(y)dy$

$= 2[y^3\, J_3(y) + c]$

$= 2\left[x^{\frac{3}{2}}\, J_3(\sqrt{x}) + c\right]$